FUNCTIONS AND GRAPHS

FLASHBACK: Revising Grade 10 Functions

Straight line graphs

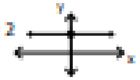
$y = mx + c$ —— y-intercept

Gradient: 1) $\frac{y_2 - y_1}{x_2 - x_1}$

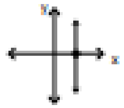
2) || lines $m_1 = m_2$

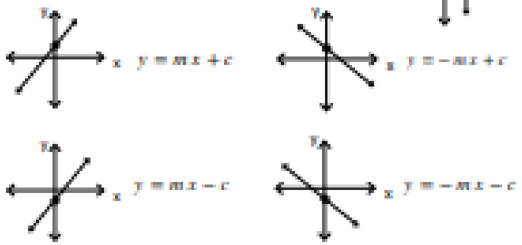
3) ⊥ lines $m_1 \times m_2 = -1$

No gradient



Undefined gradient

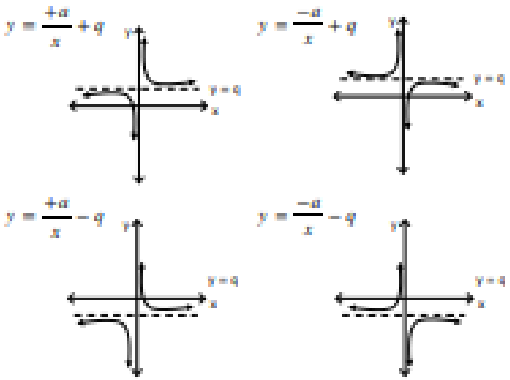




Hyperbolas

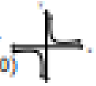
$y = \frac{a}{x} + q$ —— q = asymptote

'a' determines shape



EXAMPLE

Sketch the graph with the equation: $f(x) = \frac{2}{x} + 2$

- Shape: a > 0 ∴ 
- x-intercept (y = 0)

$0 = \frac{2}{x} + 2$

$-2 = \frac{2}{x}$

$-2x = 2$

$x = -1$

- y-intercept (x = 0)

$y = \frac{2}{0}$ → undefined

No y-intercept

- Use a table and plot at least 2 other points

x	-2	-1	0	1	2
y	1	0	~~/~~	4	3

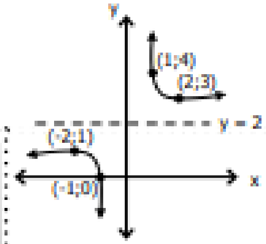
Asymptote!

- Asymptotes

$x = 0$

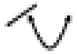
$y = 2$

NOTE: Start with the asymptotes when sketching



Parabolas (Quadratic functions)

$y = ax^2 + q$ —— y-intercept

a > 0  OR a < 0 

EXAMPLE

Sketch the graph with the equation: $f(x) = x^2 - 4$

- Shape: a > 0 ∴ ∪
- x-intercept (y = 0)

$0 = x^2 - 4$

Option 1: Factorise

$0 = (x+2)(x-2)$

$x = -2$ OR $x = 2$

Option 2: Solve for x

$0 = x^2 - 4$

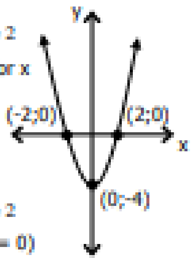
$4 = x^2$

$\pm\sqrt{4} = x$

$x = -2$ OR $x = 2$

- y-intercept (x = 0)

$y = -4$

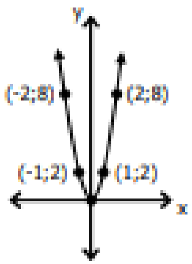


EXAMPLE

Sketch the graph with the equation: $f(x) = 2x^2$

- Shape: a > 0 ∴ ∪
- Use a table (on your calculator)

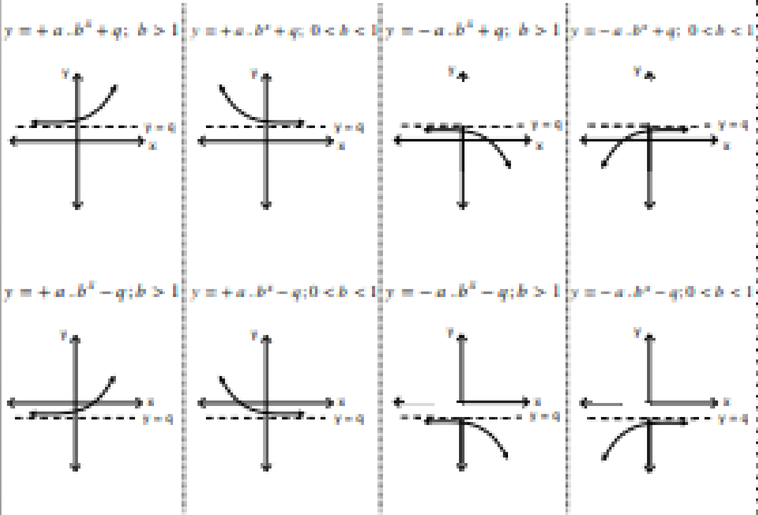
x	-2	-1	0	1	2
y	8	2	0	2	8



Exponential Graphs

$y = a^x + q$ OR $y = a.(b^x) + q$ q = asymptote

'a' determines shape



EXAMPLE

Sketch the graph with the equation $f(x) = 2^x - 1$

- Shape: a > 0 ∴ 
- x-intercept (y = 0)

$0 = 2^x - 1$

$1 = 2^x$

$2^0 = 2^x$

$0 = x$

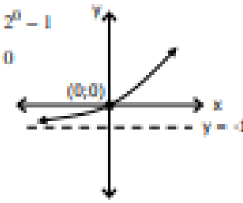
- Asymptote (y = q)

$y = -1$

- y-intercept (x = 0)

$y = 2^0 - 1$

$y = 0$



FUNCTIONS AND GRAPHS

Grade 11 Functions: Quadratic Functions

$y = a(x-p)^2 + q$

Shape:
- a > 0 ∪
- a < 0 ∩

Horizontal shift:
- x-value of turning pt
- axis of symmetry

Vertical shift:
- y-value of turning pt

Steps for sketching $y = a(x-p)^2 + q$

1. Determine the shape ('a')
2. Find the x- and y-intercepts
3. Find the turning point
4. Plot points and sketch graph

EXAMPLE 1

Sketch $f(x) = (x+1)^2 - 9$

- Shape: a > 0 ∴ ∪
- x-intercept (y = 0)

  $0 = (x+1)^2 - 9$

  $9 = (x+1)^2$

  $\pm\sqrt{9} = x + 1$

  $+3 = x + 1$ OR $-3 = x + 1$

  $2 = x$ OR $-4 = x$
- y-intercept (x = 0)

  $y = (0+1)^2 - 9$

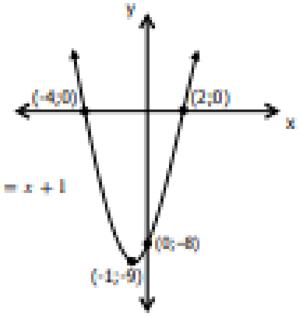
  $y = -8$
- Turning point (p;q)

  $(-1;-9)$
- Axis of symmetry

  $x = -1$
- Domain

  $x \in R$
- Range

  $y \geq -9$



Remember:
$(x-(-1))^2 - 9$
$(x-p)^2 + q$

NOTE:
$f(x) = x^2$
→ moved 1 unit to the left
→ moved 9 units down

Steps for sketching $y = ax^2 + bx + c$

1. Determine the shape ('a')
2. Find the x- and y-intercepts
3. Find the turning point $(\frac{-b}{2a})$
4. Plot points and sketch graph

EXAMPLE 2

Sketch $f(x) = x^2 - 4x + 3$

- Shape: a > 0 ∴ ∪
- x-intercept (y = 0)

Option 1:

$0 = x^2 - 4x + 3$

$0 = (x-3)(x-1)$

$x = 3$ OR $x = 1$

Option 2:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)}$

$x = \frac{4 \pm 2}{2}$

$x = 3$ OR $x = 1$

- y-intercept (x = 0)

  $y = 3$
- Turning point (p;q)

  1) x-value of TP $= \frac{-b}{2a} = \frac{-(-4)}{2(1)}$

  $x = 2$

  2) Subst. into original eq:

  $y = (2)^2 - 4(2) + 3$

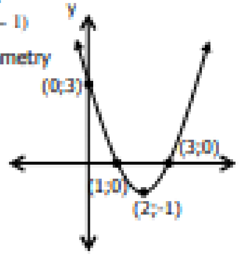
  $y = -1$

  TP $(2;-1)$
- Axis of symmetry

  $x = 2$
- Domain

  $x \in R$
- Range

  $y \geq -1$



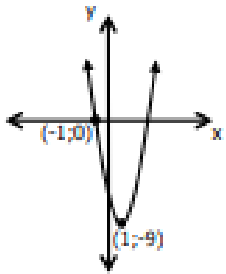
Finding the equation in the form $y = a(x-p)^2 + q$

Given the **turning point** and **another point**

1. Substitute the turning point into $y = a(x-p)^2 + q$
2. Substitute the other point into the equation to find 'a'
3. Determine the equation of the graph

EXAMPLE 1

Find the equation of the following graph:



- Turning point (p;q)

  $p = 1$ and $q = -9$

  $y = a(x-1)^2 - 9$
- Other point

  $(-1;0)$

  $0 = a(-1-1)^2 - 9$

  $9 = 4a$

  $a = \frac{9}{4}$
- Equation

  $y = \frac{9}{4}(x-1)^2 - 9$

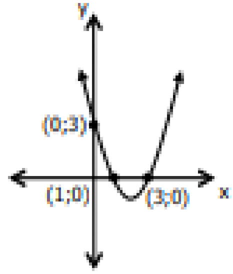
Finding the equation in the form $y = ax^2 + bx + c$

Given the **x-intercepts** and **another point**

1. Substitute the x-intercepts into $y = a(x-x_1)(x-x_2)$
2. Substitute the other point in to find 'a'
3. Write/simplify your final equation

EXAMPLE 2

Find the equation of the following graph:



- x-intercepts

  $x = 1$ OR $x = 3$

  Formula: $y = a(x-x_1)(x-x_2)$

  $y = a(x-1)(x-3)$
- Other point

  $(0;3)$

  $3 = a(-1)(-3)$

  $1 = a$
- Equation

  $y = 1(x-1)(x-3)$

  $y = x^2 - 4x + 3$

NOTE:
If you need to write this equation in the form $y = a(x-p)^2 + q$ complete the square

$y = (x-2)^2 + 3 - 4$

$y = (x-2)^2 - 1$

FUNCTIONS AND GRAPHS

Exponential Graphs

$y = ab^{x-p} + q$

Shape:
- b > 1
- 0 < b < 1

Vertical shift:
- asymptote

Steps for sketching $y = ab^{x-p} + q$

1. Determine the asymptote ('q')
1. Determine the shape ('a')
2. Find the x- and y-intercepts
3. Plot points (at least 2 others) and sketch graph

EXAMPLE 1

Sketch $f(x) = 2^{x+1} + 1$

- Asymptote

  $y = 1$
- Shape: a > 0 ∴

- x-intercept (y = 0)

  $0 = 2^{x+1} + 1$

  $-1 = 2^{x+1}$

  Not possible to solve for x

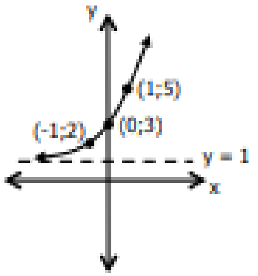
  ∴ No x-intercept
- y-intercept (x = 0)

  $y = 2^{0+1} + 1$

  $y = 3$
- Domain

  $x \in R$
- Range

  $y > 1$



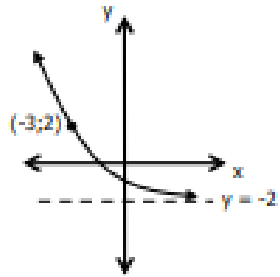
Finding the equation in the form $y = ab^{x-p} + q$

Given the **asymptote** and **another point**

1. Substitute the asymptote into the equation
2. Substitute the other point in
3. Write/simplify your final equation

EXAMPLE 2

Find the equation of the following graph given $y = b^{x+1} + q$:



- Asymptote

  $q = -2$

  $y = b^{x+1} - 2$
- Other point

  $(-3;2)$

  $2 = b^{-3+1} - 2$

  $4 = b^{-2}$

  $4 = \frac{1}{b^2}$

  $b^2 = \frac{1}{4}$

  $b = \pm\frac{1}{2}$ $\qquad b \neq -\frac{1}{2}$

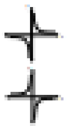
  $b = +\frac{1}{2}$
- Equation

  $y = (\frac{1}{2})^{x+1} - 2$

Hyperbola

$y = \frac{a}{x-p} + q$

Shape:
- a > 0
- a < 0



Horizontal shift:
- asymptote

Vertical shift:
- asymptote

Steps for sketching $y = \frac{a}{x-p} + q$

1. Determine the asymptotes (y = 'q' and x = 'p')
1. Determine the shape ('a')
2. Find the x- and y-intercepts
3. Plot points (at least 2 others) and sketch graph

EXAMPLE 1

Sketch $f(x) = \frac{-1}{x-2} - 1$

- Asymptotes

  $x = 2$

  $y = -1$
- Shape: a < 0 ∴

- x-intercept (y = 0)

  $0 = \frac{-1}{x-2} - 1$

  $1 = \frac{-1}{x-2}$

  $x - 2 = -1$

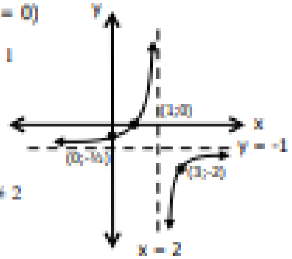
  $x = 1$
- y-intercept (x = 0)

  $y = \frac{-1}{-2} - 1$

  $y = -\frac{1}{2}$
- Domain

  $x \in R; x \neq 2$
- Range

  $y \in R; y \neq -1$



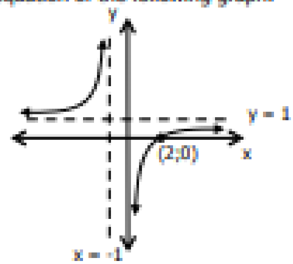
Finding the equation in the form $y = \frac{a}{x-p} + q$

Given the **asymptotes** and **another point**

1. Substitute the asymptotes into the equation
2. Substitute the other point into the equation to find 'a'
3. Write/simplify your final equation

EXAMPLE 2

Find the equation of the following graph:



- Asymptote

  $y = 1$ and $x = -1$

  $f(x) = \frac{a}{x-(-1)} + 1$

  $f(x) = \frac{a}{x+1} + 1$
- Other point

  $(2;0)$

  $0 = \frac{a}{2+1} + 1$

  $-1 = \frac{a}{3}$

  $-3 = a$
- Equation

  $f(x) = \frac{-3}{x+1} + 1$

Lines of Symmetry:

Use point of intersection of asymptotes. (-1;1)

$y = x + c$	$(-1;1)$	$y = -x + c$	$(-1;1)$
$1 = -1 + c$		$1 = 1 + c$	
$2 = c$		$0 = c$	
$y = x + 2$		$y = -x$	

FUNCTIONS AND GRAPHS

Deductions from Graphs

DISTANCE

Steps for determining VERTICAL DISTANCE

1. Determine the vertical distance
   Vertical distance = top graph – (bottom graph)
2. Substitute the given x-value to derive your answer

Steps for determining HORIZONTAL DISTANCE

1. Find the applicable x-values
   $AB = x_B - x_A$ (largest – smallest)

Steps for determining MAXIMUM DISTANCE

1. Determine the vertical distance
   Vertical distance = top graph – bottom graph
2. Complete the square
   $y = a(x-p)^2 + q$
3. State the maximum distance
   $y = a(x-p)^2 + (\mathbf{q})$ → q is the max distance

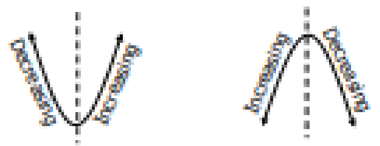
> **NOTE:**
> - Distance is always positive
> - Distance on a graph is measured in units

INTERSECTION OF GRAPHS

Steps for determining POINTS OF INTERCEPTION

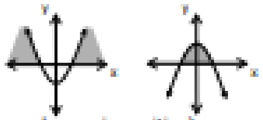
1. Equate the two functions
   $f(x) = g(x)$
2. Solve for x (look for the applicable x-value: A or B)
3. Substitute the applicable x-value into any of the two equations to find 'y'

INCREASING/DECREASING



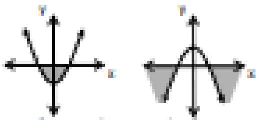
NOTATION

- $f(x) > 0$ ⊕
  (above the line y = 0)



(i.e. where y is positive)

- $f(x) < 0$ ⊖
  (below the line y = 0)

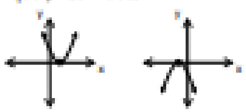


(i.e. where y is negative)

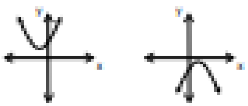
- ⊕ ⊖
  $f(x) \cdot g(x) \le 0$ ⊖
  ⊖ ⊕
  (one graph lies above y = 0 and one graph lies below y = 0)
- $f(x) \ge g(x)$
  top bottom
  (i.e. f(x) lies above g(x))
- $f(x) = g(x)$
  (point of intersection)

ROOTS & PARABOLAS

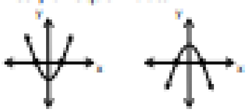
- Equal, real roots



- Non-real/ No real roots

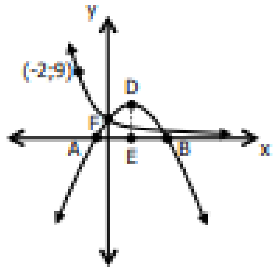


- Real, unequal roots



EXAMPLE 1

$f(x) = ax^2 + bx + c$ and $g(x) = k^x$ are sketched. D is the turning point of f(x) with the axis of symmetry at x=2. AB is 6 units.



Questions:

a. Determine the value of k.

b. Determine the x-values of A and B.

c. Show that $a = \frac{-1}{5}$ and $b = \frac{4}{5}$.

d. Determine the coordinates of D.

e. Determine the maximum distance of DE.

f. Determine the values of p for which:
$\frac{-1}{5}x^2 + \frac{4}{5}x + p < 0$

g. Determine for which values of x:
  i. $f(x) \ge 0$
  ii. $\frac{f(x)}{g(x)} > 0$
  iii. $f(x)$ is increasing

Solutions:

a. $(-2;9)$
$9 = k^{-2}$
$9 = \frac{1}{k^2}$
$k = \pm\frac{1}{3}$
$k = \frac{1}{3}$

b. $E = (2;0)$ and $AB = 6\ units$
$A = (-1;0)\quad x = -1$
$B = (5;0)\quad x = 5$

c. $y = a(x - x_1)(x - x_2)$
$(-1;0)$ and $(5;0)$
$y = a(x+1)(x-5)$
Use $F(0;1)$
$1 = a(+1)(-5)$
$-\frac{1}{5} = a$
$y = -\frac{1}{5}(x^2 - 4x + 1)$
$y = -\frac{1}{5}x^2 + \frac{4}{5}x + 1$
$b = \frac{4}{5}$

d. $y = -\frac{1}{5}(2)^2 + \frac{4}{5}(2) + 1$
$y = \frac{9}{5}\quad \therefore\ D = (2;\frac{9}{5})$

e. $\frac{9}{5}$ units (y-value of co-ordinate D is also TP)

f. $-\frac{1}{5}x^2 + \frac{4}{5}x + p < 0$ ⊖
$p < -\frac{9}{5}$

> **NOTE:**
> - Interpret question as: How many units must the graph move for the max. value to be < 0

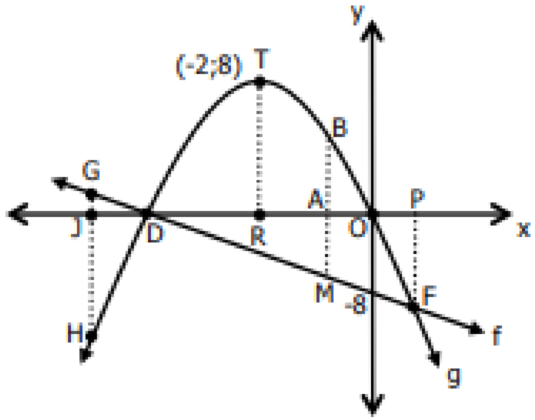
g.
  i. $x \in [-1;5]$
  ii. $x \in (-1;5)$
  iii. $x \in (-\infty;2)$

FUNCTIONS AND GRAPHS

Deductions from Graphs

EXAMPLE 2

$f(x) = mx + c$ and $g(x) = a(x - p)^2 + q$ are sketched below. T is the turning point of g(x).



<u>Questions:</u>

Determine:

a. The value of a, p, q, m and c.

b. The length of OD.

c. The length of TR.

d. The equation of TR.

e. BM if OA = 1 unit.

f. OJ if GH = 28 units.

g. The length of FP.

h. The maximum length of BM.

i. The value of k for which $-2x^2 - 8x + k$ has two equal roots.

j. For which value(s) of x will $\frac{f(x)}{g(x)} < 0$?

<u>Solutions:</u>

a. $y = a(x + 2)^2 + 8 \quad (0;0)$

$-8 = 4a$

$-2 = a$ and $p = -2$ and $q = 8$

$\therefore \; g(x) = -2(x + 2)^2 + 8$

$D\ (-4;0)$

$\therefore \; m = \frac{8}{-4} = -2$ and $c = -8$

b. OD = 4 units

c. TR = 8 units

d. TR: $x = -2$

e. $g(x) = -2(x + 2)^2 + 8$

$= -2(x^2 + 4x + 4) + 8$

$= -2x - 8x - 8 + 8$

$= -2x^2 - 8x$

BM = $g(x) - f(x)$

BM $= -2x^2 - 8x - (-2x - 8)$

$= -2x^2 - 6x + 8$

BM $= -2(-1)^2 - 6(-1) + 8$

BM = 12 units

f. $28 = -2x - 8 - (-2x^2 - 8x)$

$28 = -2x - 8 + 2x^2 + 8x$

$0 = 2x^2 + 6x - 36$

$0 = 2(x^2 + 3x - 18)$

$0 = 2(x + 6)(x - 3)$

$x = -6$ or $x = 3$ (NA)

$\therefore$ OJ = 6 units

g. $-2x^2 - 8x = -2x - 8$

$0 = 2x^2 + 6x - 8$

$0 = 2(x - 1)(x + 4)$

$x = 1$ or $x = -4$ (NA)

$y = -2(1) - 8$

$y = -10$

$\therefore$ FP = 10 units

h. Max length is given by TP of parabola ($L(x)$) given by $L(x) = g(x) - f(x)$. Find the TP by completing the square.

$\therefore$ Max BM = $g(x) - f(x)$

$= -2x^2 - 8x - (-2x - 8)$

$= -2x^2 - 6x + 8$

$= -2(x^2 + 3x - 4)$

$= -2[(x + \frac{3}{2})^2 - 4 - \frac{9}{4}]$

$= -2[(x + \frac{3}{2})^2 - \frac{25}{4}]$

$= -2(x + \frac{3}{2})^2 + \frac{25}{2}$

$\therefore$ Max of BM = $\frac{25}{2}$ units

i. k = -8

j. $x \in (-\infty;0);\ x \neq -4$