Gr 11 TECHNICAL MATHEMATICS ALGEBRA

PROVE THE NATURE OF THE ROOTS

The nature of the roots will be supplied and the discriminant can be used to prove the nature, with either one, or no, unknown value.

Steps to prove the nature of roots (NO unknown):

1. Put the equation in its standard form
2. Substitute the correct values in and calculate the discriminant
3. Determine the roots and confirm whether they are as supplied

EXAMPLE

Prove the equation has two, unequal, irrational roots:
$x^2 = 2x + 9$

1. Standard form

$x^2 - 2x - 9 = 0$

a b c

2. Calculate the discriminant

$\Delta = b^2 - 4ac$
$\Delta = (-2)^2 - 4(1)(-9)$
$\Delta = 4 + 36$
$\Delta = 40$

3. Determine the roots

The Roots are:
Real ($\Delta > 0$)
Unequal ($\Delta \neq 0$)
Irrational ($\Delta \neq$ perfect square)

Steps to prove the nature of roots (ONE unknown):

1. Put the equation in its standard form
2. Substitute the correct values in and calculate the discriminant
3. Determine the roots and confirm whether they are as supplied

EXAMPLE

For the equation $x(6x - 7m) = 5m^2$, prove that the roots are real, rational and unequal if $m > 0$

1. Standard form

$6x^2 - 7mx - 5m^2 = 0$

a b c

2. Calculate the discriminant

$\Delta = b^2 - 4ac$
$\Delta = (-7m)^2 - 4(6)(-5m^2)$
$\Delta = 49m^2 + 120m^2$
$\Delta = 169m^2$

3. Determine the roots

The Roots are:
Real ($\Delta > 0$)
Unequal ($\Delta \neq 0$)
Rational ($\Delta =$ perfect square)

DETERMINING THE NATURE OF ROOTS WITHOUT SOLVING THE EQUATION

The roots of an equation can be determined by calculating the value of the discriminant (Δ).

Steps to determine the roots using the discriminant:

1. Put the equation in its standard form
2. Substitute the correct values in and calculate the discriminant
3. Determine the nature of the roots of the equation

EXAMPLE

Determine the nature of the roots of $x^2 = 2x + 1$ without solving the equation

1. Standard form

$x^2 = 2x + 1$
$x^2 - 2x - 1 = 0$

(a, b, c)

2. Calculate the discriminant

$\Delta = b^2 - 4ac$
$\Delta = (-2)^2 - 4(1)(-1)$
$\Delta = 4 + 4$
$\Delta = 8$

3. Determine the nature of the roots

The Roots are:
Real ($\Delta > 0$)
Unequal ($\Delta \neq 0$)
Irrational ($\Delta \neq$ perfect square)

FOR WHICH VALUES OF k WILL THE EQUATION HAVE EQUAL ROOTS?

The discriminant (Δ) can be used to calculate the unknown value of k. (e.g. Ask yourself, for which values of k will the discriminant be 0?)

Steps to determine the values of k using the discriminant:

1. Put the equation in its standard form
2. Substitute the correct values in and calculate the discriminant
3. Equate the discriminant to 0 and solve for k (quadratic equation)

EXAMPLE

For which values of k the equation will have equal roots?

REMEMBER: $\Delta = 0$ for equal roots

1. Standard form

$x^2 + 2kx = -4x - 9k$
$x^2 + \underline{2kx + 4x} + 9k = 0$

(a, b, c)

2. Calculate the discriminant

$\Delta = b^2 - 4ac$
$\Delta = (2k + 4)^2 - 4(1)(9k)$
$\Delta = 4k^2 + 16k + 16 - 36k$
$\Delta = 4k^2 - 20k + 16$

3. Equate to zero (0) and solve for k

$0 = 4k^2 - 20k + 16 \quad (\div 4)$
$0 = k^2 - 5k + 4$
$0 = (k - 1)(k - 4)$
Therefore $k = 1$ **or** $k = 4$
k needs to either be **1** or **4** to ensure that the discriminant of the equation is 0 (the discriminant **must** be 0 in order for equal roots)

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QUADRATIC EQUATIONS

SOLVING QUADRATIC EQUATIONS

FACTORING	QUADRATIC FORMULA	DIFFERENCE OF TWO SQUARES	COMPLETE THE SQUARE	ONE ROOT	TWO ROOTS	FRACTIONS AND RESTRICTIONS
1. Put in standard form 2. Apply the zero factor law 3. List possible solutions	Substitute into the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where a = coefficient of x^2, b = coefficient of x, c = constant term		1. Write in standard form 2. Move C across 3. Divide both sides by A 4. Add (½ x b)^2 to both sides 5. Factorise and solve	1. Substitute the known root 2. Solve for the variable 3. Substitute the value of the variable and solve for the root	1. Substitute the roots into the equation 2. Use "FOIL" for the quadratic equation	1. Find the LCD and list restrictions 2. Solve for x 3. Check your answers against your restrictions REMEMBER: $\frac{0}{x} = 0$ BUT $\frac{x}{0} = undefined$
Eg. $x^2 = -2x + 63$ $x^2 + 2x - 63 = 0$ Find factors of 63 so that F1 x F2 = –63 and F1 + F2 = 2 $(x + 9)(x - 7) = 0$ $x + 9 = 0$ or $x - 7 = 0$ $x = -9$ $x = 7$	Eg. $-3x^2 = -12 + 7x$ $-3x^2 - 7x + 12 = 0$ a = –3; b = –7; c = 12 $x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(-3)(12)}}{2(-3)}$ $x = \frac{7 \pm \sqrt{49 + 144}}{-6}$ $x = \frac{7 \pm \sqrt{193}}{-6}$ $x = \frac{7 + \sqrt{193}}{-6}$ or $x = \frac{7 - \sqrt{193}}{-6}$ Answer in surd form or can be calculated/rounded off to 2 decimals x = –3,48 OR x = 1,15	Either method may be used Eg. $x^2 = 25$ $x^2 - 25 = 0$ $(x - 5)(x + 5) = 0$ $x = 5$ or $x = -5$ Therefore $x = \pm 5$ $\sqrt{x^2} = \pm\sqrt{25}$ $x = \pm 5$	Eg. $x^2 + 2x = 1$ (½ . 2)^2 (1) $x^2 + 2x + 1 = 1 + 1$ $\sqrt{(x+1)^2} = \pm\sqrt{2}$ $x + 1 = \pm\sqrt{2}$ $x + 1 = -\sqrt{2}$ or $x + 1 = \sqrt{2}$ $x = -1 - \sqrt{2}$ $x = -1 + \sqrt{2}$	Eg. $x^2 + mx - 15 = 0$, where 5 is a root. $(5)^2 + (5)m - 15 = 0$ $25 + 5m - 15 = 0$ $5m = -10$ $m = -2$ $x^2 - 2x - 15 = 0$ $(x - 5)(x + 3) = 0$ $x = 5$ or $x = -3$ (given)	Eg. –9 and 7 are the roots of a quadratic equation $x = -9$ or $x = 7$ $x + 9 = 0$ $x - 7 = 0$ $(x + 9)(x - 7) = 0$ $x^2 - 7x + 9x - 63 = 0$ $x^2 + 2x - 63 = 0$	Eg. $\frac{x}{x-2} = \frac{1}{x-3} - \frac{2}{2-x}$ $\frac{x}{x-2} = \frac{1}{x-3} + \frac{2}{x-2}$ LCD: $(x-2)(x-3)$ Restrictions: $x - 2 \neq 0$; $x - 3 \neq 0$ $\frac{x\cancel{(x-2)}(x-3)}{\cancel{(x-2)}} = \frac{1(x-2)\cancel{(x-3)}}{\cancel{(x-3)}} + \frac{2\cancel{(x-2)}(x-3)}{\cancel{(x-2)}}$ $x(x - 3) = 1(x - 2) + 2(x - 3)$ $x^2 - 3x = x - 2 + 2x - 6$ $x^2 - 3x - x - 2x + 2 + 6 = 0$ $x^2 - 6x + 8 = 0$ $(x - 4)(x - 2) = 0$ $x - 4 = 0$ or $x - 2 = 0$ $x = 4$ $x = 2$ Check restrictions: $x \neq 2$, $x \neq 3$ Thus, x = 4 is the only solution.

RATIONALISING THE DENOMINATOR

EXAMPLE 1

Express the following with rational denominators:

1. $\frac{3}{\sqrt{7}}$

$= \frac{3}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$

$= \frac{3\sqrt{7}}{7}$

2. $\frac{6+3\sqrt{2}}{2\sqrt{3}}$

$= \frac{6+3\sqrt{2}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{6\sqrt{3}+3\sqrt{6}}{2 \times 3}$

$= \frac{6\sqrt{3}+3\sqrt{6}}{6}$

$= \frac{2\sqrt{3}+\sqrt{6}}{2}$

EXAMPLE 2

If $x = \sqrt{3}+2$, simplify: $\frac{x^2+2}{x-2}$ and express the answer with a rational denominator

1. $\frac{x^2+2}{x-2}$

$= \frac{(\sqrt{3}+2)^2+2}{(\sqrt{3}+2)-2}$

$= \frac{3+4\sqrt{3}+4+2}{\sqrt{3}}$

$= \frac{9+4\sqrt{3}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{9\sqrt{3}+4 \cdot 3}{3}$

$= 3\sqrt{3}+4$

Why do we do this?

Multiplying the binomial by itself will give us a trinomial with an irrational middle term. To avoid this, we multiply the binomial by its **conjugate** (same binomial with the opposite sign) to create a difference of two squares.

EXAMPLE 1

Express the following fractions with rational denominators:

1. $\frac{3}{5-\sqrt{7}}$

$= \frac{3}{5-\sqrt{7}} \times \frac{5+\sqrt{7}}{5+\sqrt{7}}$

$= \frac{15+3\sqrt{7}}{25-7}$

$= \frac{15+3\sqrt{7}}{18}$

$= \frac{5+\sqrt{7}}{6}$

2. $\frac{7}{\sqrt{x}-\frac{1}{\sqrt{x}}}$

$= \frac{7}{\sqrt{x}-\frac{1}{\sqrt{x}}} \times \frac{\sqrt{x}+\frac{1}{\sqrt{x}}}{\sqrt{x}+\frac{1}{\sqrt{x}}}$

$= \frac{7\sqrt{x}+\frac{7}{\sqrt{x}}}{x-\frac{1}{x}}$

$= \frac{\frac{7x+7}{\sqrt{x}}}{\frac{x^2-1}{x}}$

$= \frac{7x+7}{\sqrt{x}} \div \frac{x^2-1}{x}$

$= \frac{7\cancel{(x+1)}}{\sqrt{x}} \times \frac{x}{\cancel{(x+1)}(x-1)}$

$= \frac{7x}{\sqrt{x}(x-1)} \times \frac{\sqrt{x}}{\sqrt{x}}$

$= \frac{7\cancel{x}\sqrt{x}}{\cancel{x}(x-1)}$

$= \frac{7\sqrt{x}}{(x-1)}$

5. Exponential Equations:

Make sure that you get a term on the one side of the equation that has a base that is equal to the base with the unknown exponent. Then, drop the bases, equate the exponents and solve.

Hints:

- NEVER drop the base if the terms are separated by a + or –
  - Remove common factors until the equation is in its simplest form and then solve
- Always convert decimals to fractions and then to bases with negative exponents

EXAMPLES

1. $4^x = 8$

$2^{2x} = 2^3$

$2x = 3$

$x = \frac{3}{2}$

2. $0{,}0625^x = 64$

$\left(\frac{1}{16}\right)^x = 2^6$

$\left(\frac{1}{2^4}\right)^x = 2^6$

$2^{-4x} = 2^6$

$-4x = 6$

$x = \frac{-3}{2}$

3. $2 \cdot 3^{x+1} + 5 \cdot 3^x = 33$

$3^x(2 \cdot 3 + 5) = 33$

$3^x(11) = 33$

$3^x = 3^1$

$x = 1$

4. $27^{3x+1} = 81^{2x+5}$

$(3^3)^{3x+1} = (3^4)^{2x+5}$

$3^{9x+3} = 3^{8x+20}$

$9x + 3 = 8x + 20$

$x = 17$

5. $0{,}5^x \cdot \sqrt{1+\frac{9}{16}} = 10$

$\left(\frac{1}{2}\right)^x \cdot \sqrt{\frac{25}{16}} = 10$

$2^{-x} \cdot \frac{5}{4} = 10$

$2^{-x} = 8$

$2^{-x} = 2^3$

$-x = 3$

$x = -3$

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LAWS OF EXPONENTS

Laws of exponents only apply to multiplication, division, brackets and roots. NEVER adding or subtracting

	Algebraic Notation	Exponential Notation	Exponential Law in operation
1	$16 = 2 \times 2 \times 2 \times 2$	$16 = 2^4$	When we MULTIPLY the SAME bases we ADD the exponents.
2	$\frac{64}{16} = 4$	$\frac{2^6}{2^4} = 2^2$	When we DIVIDE the SAME bases we MINUS the exponents (always top minus bottom).
3	$4^3 = 64$	$(2^2)^3 = 2^6$	When we have the exponents outside the BRACKETS we DISTRIBUTE them into the brackets.
4	$\frac{64}{64} = 1$	$\frac{2^6}{2^6} = 2^0 = 1$	Any base to the POWER OF ZERO is equal to one. (But 0^0 is undefined).
5	$\sqrt[3]{64} = 4$	$\sqrt[3]{2^6} = 2^2$	The POWER inside the root is DIVIDED by the size of the root.
6	$4 \times 9 = 36$	$2^2 \times 3^2 = 6^2$	When we have non-identical bases, but identical exponents, we keep the exponents and multiply the bases. (This same rule will also apply for division).
7	$\sqrt{2} \times \sqrt{3} = \sqrt{6}$	$2^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 6^{\frac{1}{2}}$	
8	$\sqrt{2} \times \sqrt{2} = \sqrt{4} = 2$	$2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^1$	Any square root multiplied by itself will equal the term inside the root.

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1. Linear Equations:

Move all the variables to the one side, and the constants to the other to solve. Linear equations have only **one** solution.

EXAMPLES

Solve:

1. $3(x-2)+10=5-(x+9)$

$3x-6+10=5-x-9$

$3x+4=-x-4$

$4x=-8$

$x=-2$

2. $(x-2)^2-1=(x+3)(x-3)$

$\cancel{x^2}-4x+4-1=\cancel{x^2}-9$

$-4x+3=-9$

$-4x=-12$

$x=3$

2. Quadratic Equations:

Move everything to one side and equate to zero. By factorising the trinomial, you should find **two** solutions.

EXAMPLES

Solve: (Q3 - Q6 are the most likely exam-type questions)

1. $x^2+5=6x$

$x^2-6x+5=0$

$(x-5)(x-1)=0$

$x=5$ or $x=1$

2. $(3x-4)(5x+2)=0$

$3x=4$ or $5x=-2$

$x=\frac{4}{3}$ or $x=\frac{-2}{5}$

3. $x^4+3x^2-10=0$

$(x^2+5)(x^2-2)=0$

$x^2=-5$ or $x^2=2$

No sol. or $x=\pm\sqrt{2}$

4. $x^{\frac{2}{3}}+3x^{\frac{1}{3}}-10=0$

$(x^{\frac{1}{3}}+5)(x^{\frac{1}{3}}-2)=0$

$x^{\frac{1}{3}}=-5$ or $x^{\frac{1}{3}}=2$

$x=-125$ or $x=8$

5. $x+3\sqrt{x}-10=0$

$x+3x^{\frac{1}{2}}-10=0$

$(x^{\frac{1}{2}}+5)(x^{\frac{1}{2}}-2)=0$

$x^{\frac{1}{2}}=-5$ or $x^{\frac{1}{2}}=2$

$\sqrt{x}=-5$ or $\sqrt{x}=2$

No sol. or $x=4$

6. $2^{2x}-6\cdot 2^x-16=0$

$(2^x+2)(2^x-8)=0$

$2^x=-2$ or $2^x=8$

No sol. or $2^x=2^3$

$x=3$

EQUATIONS

3. Simultaneous Equations:

Solve for two unknowns in two different equations using the substitution method. Remember to solve for both unknowns by substituting them back into the original equation.

EXAMPLES

Solve:

1. Equation 1: $2x+3y=18$

Equation 2: $-3x+5y=11$

From 1: $2x+3y=18$

$2x=-3y+18$

$x=\frac{-3y+18}{2}$ 1a

Sub 1a into 2: $-3x+5y=11$

$-3\left(\frac{-3y+18}{2}\right)+5y=11$

$\frac{9y-54}{2}+5y=11$

$9y-54+10y=22$

$19y=76$

$y=4$....3

Sub 3 into 1: $2x+3(4)=18$

$2x=6$

$x=3$

$(3;4)$

2. Equation 1: $y+3x=2$

Equation 2: $y^2-9x^2=16$

From 1: $y+3x=2$

$y=-3x+2$...1a

Sub 1a into 2: $y^2-9x^2=16$

$(-3x+2)^2-9x^2=16$

$\cancel{9x^2}-12x+4-\cancel{9x^2}=16$

$-12x=12$

$x=-1$...3

Sub 3 into 1: $y+3(-1)=2$

$y=5$

$(-1;5)$